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On the boundary coupling of topological Landau-Ginzburg models

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Abstract: I propose a general form for the boundary coupling of B-type topological Landau-Ginzburg models. In particular, I show that the relevant background in the open string sector is a (generally non-abelian) superconnection of type $(0, 1)$ living in a complex superbundle defined on the target space, which I allow to be a non-compact Calabi-Yau manifold. This extends and clarifies previous proposals. Generalizing an argument due to Witten, I show that BRST invariance of the partition function on the worldsheet amounts to the condition that the $(0, \leq 2)$ part of the superconnection's curvature equals a constant endomorphism plus the Landau-Ginzburg potential times the identity section of the underlying superbundle. This provides the target space equations of motion for the open topological model.

Keywords: Topological Field Theories, Topological Strings, D-branes.

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Contents

1. Introduction

It was recently pointed out [1−6] that Landau-Ginzburg models contain many more topological B-type branes than previously believed. This is potentially important since it seems to allow for a simple realization of the framework of [7−17] in such theories.

The physics argument of [2, 4−6] involves the construction of a boundary action for the untwisted (i.e. $N = 2$) model, whose B-type supersymmetry variation is shown to cancel the boundary term produced by the variation of the bulk action. The object described by this coupling can be understood as a composite of two elementary branes obtained by condensing a tachyon. More precisely, the Landau-Ginzburg model reduces to an $N = 2$ sigma model when one turns off the Landau Ginzburg potential W. Starting with two top rank B-branes of the sigma model and turning on W forces the condensation of a tachyon. This leads to a D-brane composite modeled by the boundary action considered in [2, 4−6]. Thus the basic branes of Landau-Ginzburg models are composites of elementary branes of the associated *sigma model*, with tachyon condensation driven by the potential W! This point of view leads to novel insight into the D-brane dynamics of such theories [18].

Unfortunately, the construction of the boundary coupling has been carried out only under simplifying assumptions, which must be removed in order to make further progress. Among the limitations of previous analyses are the condition that the target space of the model is an affine space $\mathbb{C}^n$, the fact that the boundary action is known only when both underlying sigma model branes have multiplicity one (i.e. they carry *abelian* gauge connections on their worldvolumes), and the fact that gauge degrees of freedom on these branes are not included. Finally, the boundary action is constructed in the untwisted, $N = 2$ model, which makes it difficult to apply off-shell techniques or perform a systematic study of localization.

In the present note, I remove each of these limitations by proposing a general boundary coupling for the topological Landau-Ginzburg model with an arbitrary noncompact Calabi-Yau target X. As it turns out, the relevant open string background is a $(0,1)$ superconnection in a complex superbundle E over X. This matches expectations from string field theory and known results from the B-twisted sigma model [19, 12, 20]. By extending an argument of [19] to the case of superconnections, I show that BRST invariance of the worldsheet partition function is restored by this coupling provided that the $(0, \leq 2)$ part of the superconnection's curvature equals a constant endomorphism plus W times the identity endomorphism of E. This provides the target space equations of motion of the topological open string background. The construction of the present paper is carried out directly in the twisted model. This leads to a few simplifications, and in particular allows us to avoid the introduction of certain complex conjugate terms.

Observation. Strictly speaking, our discussion will be classical on the worldsheet, so we could allow X to be only Kähler. However anomaly cancellation requires that X is Calabi-Yau in order for the theory to make sense at the quantum level [21].

2. The bulk action and the topological Warner term

The general formulation of closed B-type topological Landau-Ginzburg models was given a while ago in [22] by extending the work of [23]. The target space of such models is a Calabi-Yau manifold X, and the Landau-Ginzburg potential is a holomorphic function $W \in H^0(\mathcal{O}_X)$. Since any holomorphic function on a compact complex manifold is constant, we must assume that X is non-compact in order to obtain interesting models. As for the B-twisted sigma model [21], the Grassmann even worldsheet fields of the on-shell formulation are given by the components ϕ^i, $\phi^{\bar{i}}$ of the map $\phi : \Sigma \to X$, while the G-odd fields are sections η, θ and ρ of the bundles $\phi^*(\bar{T}X)$, $\phi^*(T^*X)$ and $\phi^*(TX) \otimes \mathcal{T}^*\Sigma$ over the worldsheet Σ. Here $\mathcal{T}^*\Sigma$ is the complexified cotangent bundle to Σ, while TX and $\bar{T}X$ are the holomorphic and antiholomorphic components of the complexified tangent bundle $\mathcal{T}X$ to X.

To write the bulk topological action, we introduce new fields $\chi, \bar{\chi} \in \Gamma(\Sigma, \phi^*(\bar{T}X))$ by the relations:

$$\eta^{\bar{i}} = \chi^{\bar{i}} + \bar{\chi}^{\bar{i}} \tag{2.1}$$

$$\theta_i = G_{i\bar{j}}(\chi^{\bar{j}} - \bar{\chi}^{\bar{j}})\,. \tag{2.2}$$

We will also use the quantity $\theta^{\bar{i}} = G^{\bar{i}j}\theta_j$.

As explained in [22], it is convenient to use an off-shell realization of the BRST symmetry. For this, we introduce an auxiliary G-even field $\tilde{F}$ which transforms as a section of $\phi^*(\mathcal{T}X)$. Then the BRST transformations are:

$$\delta\phi^i = 0\,, \qquad \delta\phi^{\bar{i}} = \chi^{\bar{i}} + \bar{\chi}^{\bar{i}} = \eta^{\bar{i}}$$

$$\begin{aligned}
\delta\chi^{\bar{i}} &= \tilde{F}^{\bar{i}} - \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{\chi}^{\bar{j}}\chi^{\bar{k}}\,, \qquad \delta\bar{\chi}^{\bar{i}} = -\tilde{F}^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\bar{\chi}^{\bar{j}}\chi^{\bar{k}} \\
\delta\rho^i_\alpha &= 2\partial_\alpha\phi^i \\
\delta\tilde{F}^i &= i\varepsilon^{\alpha\beta}\left[D_\alpha\rho^i_\beta + \frac{1}{4}R^i_{j\bar{l}k}(\chi^{\bar{l}}+\bar{\chi}^{\bar{l}})\rho^j_\alpha\rho^k_\beta\right]\,, \qquad \delta\tilde{F}^{\bar{i}} = \Gamma^{\bar{i}}_{\bar{j}\bar{k}}\tilde{F}^{\bar{j}}(\chi^{\bar{k}}+\bar{\chi}^{\bar{k}})\,.
\end{aligned} \tag{2.3}$$

Notice that the off-shell BRST transformations are independent of W. Moreover, the transformations of ϕ, η and ρ do not involve the auxiliary fields. In particular, we have $\delta\eta^{\bar{i}} = 0$. These observations will be used in section 4.

Let us choose a riemannian metric g on the worldsheet. Then the bulk action of [22] is:

$$S_{\text{bulk}} = S_B + S_W \tag{2.4}$$

where:

$$\begin{aligned}
S_B = \int_\Sigma d^2\sigma\sqrt{g}\Big[G_{i\bar{j}}\Big(&g^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}} - i\varepsilon^{\alpha\beta}\partial_\alpha\phi^i\partial_\beta\phi^{\bar{j}} - \frac{1}{2}g^{\alpha\beta}\rho^i_\alpha D_\beta\eta^{\bar{j}} - \\
&- \frac{i}{2}\varepsilon^{\alpha\beta}\rho^i_\alpha D_\beta\theta^{\bar{j}} - \tilde{F}^i\tilde{F}^{\bar{j}}\Big) + \frac{i}{4}\varepsilon^{\alpha\beta}R_{i\bar{l}k\bar{j}}\rho^i_\alpha\bar{\chi}^{\bar{l}}\rho^k_\beta\chi^{\bar{j}}\Big]
\end{aligned} \tag{2.5}$$

is the action of the B-twisted sigma model in the form used in that reference and $S_W = S_0 + S_1$ is the potential-dependent term, with:

$$S_0 = \frac{i}{2}\int_\Sigma d^2\sigma\sqrt{g}\left[D_{\bar{i}}\partial_{\bar{j}}\bar{W}\chi^{\bar{i}}\bar{\chi}^{\bar{j}} - (\partial_{\bar{i}}\bar{W})\tilde{F}^{\bar{i}}\right] \tag{2.6}$$

$$S_1 = -\frac{i}{2}\int_\Sigma d^2\sigma\sqrt{g}\left[(\partial_i W)\tilde{F}^i + \frac{i}{4}\varepsilon^{\alpha\beta}D_i\partial_j W\rho^i_\alpha\rho^j_\beta\right]. \tag{2.7}$$

Here $\varepsilon^{\alpha\beta} = \frac{\epsilon^{\alpha\beta}}{\sqrt{g}}$ is the Levi-Civita tensor and $\epsilon^{\alpha\beta}$ the associated density. We have rescaled the Landau-Ginzburg potential W by a factor of $i/2$ with respect to the conventions of [22]. The conventions for the target space Riemann tensor and covariantized worldsheet derivative D_α are unchanged. In S_W, we separated the term depending on W from that depending on its complex conjugate $\bar{W}$.

It was noticed in [22] that the topological sigma model action in the form (2.5) is BRST exact on closed Riemann surfaces. Since in this paper we shall allow Σ to have a nonempty boundary, we must be careful with total derivative terms. Extending the computation of [22] to this case, one finds:

$$S_B + s = \delta V_B \tag{2.8}$$

where:

$$V_B := \int_\Sigma d^2\sigma\sqrt{g}G_{i\bar{j}}\left(\frac{1}{2}g^{\alpha\beta}\rho^i_\alpha\partial_\beta\phi^{\bar{j}} - \frac{i}{2}\varepsilon^{\alpha\beta}\rho^i_\alpha\partial_\beta\phi^{\bar{j}} - \tilde{F}^i\chi^{\bar{j}}\right) \tag{2.9}$$

and:

$$s := i\int_\Sigma d^2\sigma\sqrt{g}\varepsilon^{\alpha\beta}\partial_\alpha(G_{\bar{i}j}\chi^{\bar{i}}\rho^j_\beta) = i\int_\Sigma d(G_{\bar{i}j}\chi^{\bar{i}}\rho^j) \tag{2.10}$$

is a total derivative term. Since such a term does not change physics on closed Riemann surfaces, we are free to redefine the bulk topological sigma-model action by adding it to S_B:

$$\tilde{S}_B := S_B + s = \delta V_B\,. \tag{2.11}$$

Accordingly, we shall work with the modified bulk Landau-Ginzburg action:

$$\tilde{S}_{\rm bulk} = S_{\rm bulk} + s = \tilde{S}_B + S_0 + S_1\,. \tag{2.12}$$

It is easy to check that the term S_0 is also BRST exact:

$$S_0 = \delta V_0 \tag{2.13}$$

where:

$$V_0 = -\frac{1}{2}\int_\Sigma d^2\sigma\sqrt{g}\theta^{\bar{i}}\partial_{\bar{i}}\bar{W}\,. \tag{2.14}$$

Equations (2.8) and (2.13) are local, i.e. they hold for the associated Lagrange densities without requiring integration by parts. Thus both of these relations can be applied to bordered Riemann surfaces.

It is now easy to show that the BRST variation of $\tilde{S}_{\rm bulk}$ produces a boundary term:

$$\delta\tilde{S}_{\rm bulk} = \delta S_1 = \frac{1}{2}\int_{\partial\Sigma}\rho^i\partial_i W\,. \tag{2.15}$$

The presence of a non-zero right hand side in (2.15) is known as the Warner problem [24]. As we shall see below, this term induces a deformation of the target space equations of motion in the open string sector.

3. The boundary coupling

Since B-type Landau-Ginzburg models are obtained from topological sigma models with non-compact targets by adding a potential term, it is natural to look for their open string backgrounds as deformations of the backgrounds allowed in the sigma model. It is well-known [25] that B-twisted Landau-Ginzburg models do not admit interesting elementary branes of top dimension. Therefore, one should look for D-brane composites obtained by condensing fields between brane-antibrane pairs of the *sigma model*. As explained in [11, 12, 20], topological D-brane composites of the B-twisted[1] sigma model are described by a Dolbeault version of the 'graded superconnections of total degree one' defined in [26]. Such objects are generalizations of the more familiar superconnections of type $(0,1)$. Graded rather than standard superconnections appear for the sigma model because topological D-branes are $\mathbb{Z}$-graded in that context [7, 34, 12, 15, 20]. As explained in [2, 5], turning on the Landau-Ginzburg potential will generally break the integral grading to a $\mathbb{Z}_2$ group. Thus one expects that B-type Landau-Ginzburg branes are described by $(0,1)$ superconnections, a variant of the objects considered in [35]. As we explain below, this expectation is indeed realized.

[1] A similar result exists for the A-twisted sigma model. In that case, graded superconnections of total degree one in the sense of [26] are the target space fields in the diagonal sector of Fukaya's category [27 – 33].

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To be precise, we consider a complex superbundle $E = E_+ \oplus E_-$ over X, and a superconnection $\mathcal{B}$ on E. We let $r_\pm := \mathrm{rk} E_\pm$. Remember that the bundle of endomorphisms $End(E)$ has a natural $\mathbb{Z}_2$ grading, whose even and odd components are given by:

$$End_+(E) := End(E_+) \oplus End(E_-) \tag{3.1}$$
$$End_-(E) := Hom(E_+, E_-) \oplus Hom(E_-, E_+)\,. \tag{3.2}$$

Then $\mathcal{B}$ can be viewed as a section of the bundle $[\mathcal{T}^*X \otimes End_+(E)] \oplus End_-(E)$. In a local frame of E compatible with the grading, this is simply a matrix:

$$\mathcal{B} = \begin{bmatrix} A^{(+)} & F \\ G & A^{(-)} \end{bmatrix} \tag{3.3}$$

whose diagonal entries $A^{(\pm)}$ are connection one-forms on $E_\pm$, while F, G are elements of $Hom(E_-, E_+)$ and $Hom(E_+, E_-)$. As for the B-model, we require[2] that the superconnection has type $(0,1)$, i.e. the one-forms $A^{(\pm)}$ belong to $\Omega^{(0,1)}(End(E_\pm))$. The morphism F should not be confused with the curvature form used below.

When endowed with the ordinary composition of morphisms, the space of sections $\Gamma(End(E))$ becomes an associative superalgebra. The space $\mathcal{H}_0 = \Omega^{(0,*)}(End(E))$ also carries an associative superalgebra structure, which is induced from $(\Omega^{(0,*)}(X), \wedge)$ and $(\Gamma(End(E)), \circ)$ via the tensor product decomposition:

$$\Omega^{(0,*)}(End(E)) = \Omega^{(0,*)}(X) \otimes_{\Omega^{(0,0)}(X)} \Gamma(End(E))\,. \tag{3.4}$$

Note that we take form components to sit on the left. For decomposable elements $u = \omega \otimes f$ and $v = \eta \otimes g$, with homogeneous[3] ω, η and f, g, the product $\bullet$ on $\mathcal{H}_0$ takes the form:

$$u \bullet v = (-1)^{\mathrm{deg} f \ \mathrm{rk} \eta} (\omega \wedge \eta) \otimes (f \circ g)\,, \tag{3.5}$$

where deg denotes the $\mathbb{Z}_2$-valued degree in the superalgebra $End(E)$:

$$\deg(f) = 0 \quad \text{if } f \in End_+(E)\,, \qquad \deg(f) = 1 \quad \text{if } f \in End_-(E)\,. \tag{3.6}$$

The total degree on $\mathcal{H}_0$ is given by:

$$|\omega \otimes f| = \mathrm{rk}\omega + \mathrm{deg} f \pmod 2\,. \tag{3.7}$$

We also recall the supertrace on $End(E)$:

$$\mathrm{str}(f) = \mathrm{tr}\, f_{++} - \mathrm{tr}\, f_{--}\,. \tag{3.8}$$

Here $f = \left[\begin{smallmatrix} f_{++} & f_{-+} \\ f_{+-} & f_{--} \end{smallmatrix}\right]$ is an endomorphism of E with components $f_{\alpha\beta} \in Hom(E_\alpha, E_\beta)$ for $\alpha, \beta = +, -$. The supertrace has the property:

$$\mathrm{str}(f \circ g) = (-1)^{\mathrm{deg} f \mathrm{deg} g}\, \mathrm{str}(g \circ f) \tag{3.9}$$

for homogeneous elements f, g.

[2] One can remove this condition, but in that case the $(1,0)$ part of $A^{(\pm)}$ would be non-dynamical.

[3] Homogeneity of ω and η means that both are differential forms of given rank (rather than sums of forms of different ranks).

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Notice that $\mathcal{B}$ is an odd element of $\mathcal{H}_o$. Thus the twisted Dolbeault operator:

$$\bar{\mathcal{D}} = \overline{\partial} + \mathcal{B} = \begin{bmatrix} \bar{\partial} + A^{(+)} & F \\ G & \bar{\partial} + A^{(-)} \end{bmatrix} \tag{3.10}$$

induces an odd derivation $\bar{\partial} + [\mathcal{B}, \ldots]_\bullet$ of the superalgebra $(\mathcal{H}_o, \bullet)$. Here $[u, v]_\bullet := u \bullet v - (-1)^{|u||v|} v \bullet u$ is the supercommutator.

The $(0, \leq 2)$ part of the superconnection's curvature has the form:

$$\mathcal{F}^{(0,\leq 2)} = \bar{\mathcal{D}}^2 = \overline{\partial}\mathcal{B} + \frac{1}{2}[\mathcal{B}, \mathcal{B}]_\bullet = \overline{\partial}\mathcal{B} + \mathcal{B} \bullet \mathcal{B} = \begin{bmatrix} F^{(+)}_{(0,2)} + FG & \bar{\nabla} F \\ \bar{\nabla} G & F^{(-)}_{(0,2)} + GF \end{bmatrix} \tag{3.11}$$

where $F^{(\pm)}_{(0,2)}$ are the $(0, 2)$ pieces of the curvature forms $F^{(\pm)}$ of $A^{(\pm)}$ and:

$$\begin{aligned} \bar{\nabla} F &= \bar{\partial} F + A^{(+)} \bullet F + F \bullet A^{(-)} = \bar{\partial} F + A^{(+)} \circ F - F \circ A^{(-)} \\ \bar{\nabla} G &= \bar{\partial} G + A^{(-)} \bullet G + G \bullet A^{(+)} = \bar{\partial} G + A^{(-)} \circ G - G \circ A^{(+)} . \end{aligned} \tag{3.12}$$

It is convenient to introduce the notations:

$$A := A^{(+)} \oplus A^{(-)} = \begin{bmatrix} A^{(+)} & 0 \\ 0 & A^{(-)} \end{bmatrix}, \qquad D := \begin{bmatrix} 0 & F \\ G & 0 \end{bmatrix} \tag{3.13}$$

for the diagonal and off-diagonal parts of $\mathcal{B}$. Then A is an ordinary connection one-form on E (which is compatible with the grading), while D is an odd endomorphism of E. We have $\mathcal{B} = A + D$ and:

$$\mathcal{F}^{(0,\leq 2)} = F^{(0,2)} + \bar{\nabla}_A D + D^2 . \tag{3.14}$$

Here $F^{(0,2)} = F^{(+)}_{(0,2)} + F^{(-)}_{(0,2)}$ is the $(0, 2)$ part of the curvature of A and $\bar{\nabla}_A = \bar{\partial} + [A, \cdot]_\bullet$ is the Dolbeault operator twisted by A. Notice that $[A, D]_\bullet = dx^{\bar{i}}[A_{\bar{i}}, D]$, where $[\cdot, \cdot]$ denotes the usual commutator.

To couple the model to such backgrounds, we shall extend the approach used in [19] for coupling ordinary $(0, 1)$ connections to the B-twisted sigma model (for comparison, that construction is reviewed in appendix A). Namely, we define the partition function on a bordered Riemann surface Σ by the formula:

$$Z := \int \mathcal{D}[\phi]\mathcal{D}[\tilde{F}]\mathcal{D}[\theta]\mathcal{D}[\rho]\mathcal{D}[\eta] e^{-\tilde{S}_{\text{bulk}}} \mathcal{U}_1 \ldots \mathcal{U}_h \tag{3.15}$$

where h is the number of holes and the factors $\mathcal{U}_\alpha$ have the form:

$$\mathcal{U}_\alpha := \operatorname{Str} P e^{-\oint_{C_\alpha} d\tau_\alpha M} . \tag{3.16}$$

Here C_α is the boundary circle associated with the hole labeled α, while Str is the supertrace on $GL(r_+|r_-)$. The symbol $d\tau_\alpha$ stands for the length element along C_α induced by the metric on the interior of Σ. The quantity M is given by:

$$M = \begin{bmatrix} \hat{\mathcal{A}}^{(+)} + FG & \frac{1}{2}\rho_0^i \nabla_i F \\ \frac{1}{2}\rho_0^i \nabla_i G & \hat{\mathcal{A}}^{(-)} + GF \end{bmatrix}, \tag{3.17}$$

where $\rho_0^i d\tau_\alpha$ is the pull-back of ρ^i to C_α and:

$$\hat{\mathcal{A}}^{(\pm)} := A^{(\pm)}_{\bar{i}}\dot{\phi}^{\bar{i}} + \frac{1}{2}\eta^{\bar{i}}F^{(\pm)}_{\bar{i}j}\rho_0^j \tag{3.18}$$

are connections on the bundles $\mathcal{E}_\pm$ obtained by pulling back $E_\pm$ to the boundary of Σ. The dot in (3.18) stands for the derivative $\frac{d}{d\tau_\alpha}$. Notice that $\nabla_i F = \partial_i F$ and $\nabla_i G = \partial_i G$ since A is a $(0,1)$-connection.

To insure BRST invariance of (3.15), we must choose the background superconnection $\mathcal{B}$ such that:

$$\delta\mathcal{U}_\alpha = \frac{1}{2}\mathcal{U}_\alpha\int_{C_\alpha}\rho^i\partial_i W\,, \tag{3.19}$$

Then the variation of the product $\prod_\alpha \mathcal{U}_\alpha$ compensates the Warner contribution (2.15) induced from the bulk. In the next section we show by direct computation that:

$$\delta\mathcal{U}_\alpha = -\,\mathrm{Str}\left[\; I_\alpha(\delta M)Pe^{-\oint_{C_\alpha}d\tau_\alpha M}\right] \tag{3.20}$$

where:

$$\begin{aligned} I_\alpha(\delta M) = \oint_{C_\alpha} d\tau_\alpha U_\alpha^{-1}\Big[&F_{\bar{i}\bar{j}}\eta^{\bar{i}}\dot{\phi}^{\bar{j}} - \frac{1}{4}\partial_k F_{\bar{i}\bar{j}}\eta^{\bar{i}}\eta^{\bar{j}}\rho_0^k + \eta^{\bar{i}}\nabla_{\bar{i}}(D^2) - \dot{\phi}^{\bar{i}}\nabla_{\bar{i}}D0 - \\ &- \frac{1}{2}\rho_0^i\partial_i(D^2) + \frac{1}{2}\eta^{\bar{i}}\rho_0^j\partial_j\nabla_{\bar{i}}D\Big]U_\alpha\,, \end{aligned}$$

with $U_\alpha(\tau_\alpha) \in GL(r_+|r_-)$ a certain invertible operator playing the role of 'parallel transport' defined by M along C_α. Here $F_{\bar{i}\bar{j}}$ etc. are the $(0,2)$ components of the curvature of the direct sum connection A introduced in (3.13). Hence the BRST invariance conditions are:

$$F_{\bar{i}\bar{j}} = 0 \tag{3.21}$$

$$\nabla_{\bar{i}}D = 0 \tag{3.22}$$

$$\partial_i(D^2) = \partial_i W\,. \tag{3.23}$$

The first relation says that A is integrable, so it defines a complex structure on the bundle E. The second condition means that $D \in End(E)$ is holomorphic with respect to this complex structure. Finally, the last equation requires $D^2 = c + W\mathrm{id}_E$ where c is a constant endomorphism of E. Comparing with (3.14), we see that these conditions are equivalent with:

$$\mathcal{F}^{(0,\leq 2)} = c + W\mathrm{id}_E \Longleftrightarrow \bar{\mathcal{D}}^2 = c + W\mathrm{id}_E\,. \tag{3.24}$$

This is the target space equation of motion for our open string background.

In the limit $W = 0$ and with the choice $c = 0$, relation (3.24) reduces to the condition $\mathcal{F}^{(0,\leq 2)} = 0 \Leftrightarrow \bar{\mathcal{D}}^2 = 0$, which is the target space equation of motion when coupling a topological brane-antibrane pair to the B-twisted sigma model. More precisely, this is a $\mathbb{Z}_2$ reduction of the full equations of motion for that model, which involve a graded superconnection [12, 20] due to the $\mathbb{Z}$-valued nature of the topological D-brane grade in that case.

Condition (3.24) generalizes particular cases established in [2, 4, 5]. Our proof is general and in particular works in the non-abelian case.[4] Working directly with the twisted model allows us to avoid certain conjugate terms considered in [5]. Notice that we do not construct a boundary action. From the perspective of the present paper, the approach of [2, 4, 5] is recovered when restricting to the case $r_+ = r_- = 1$, since in that situation one can replace our path ordered exponentials by path integrals over pairs of new fermionic fields living on the connected components of the boundary. Such a representation seems complicated for r_+ or r_- greater than one. While interesting in its own right, it is not necessary for our purpose.

Observation As in [19], our construction does not require a modification of the BRST operator. Since the boundary coupling introduced in (3.15) involves only the bulk world-sheet fields, its BRST variation is computed by using the bulk generator (2.3).

4. BRST variation of the superholonomy factors

In this section we prove the crucial relation (3.20). For this, let us focus a given boundary component C, whose proper length coordinate we shall denote by τ. To specify such a coordinate, we must chose an origin on the circle C. While various intermediate steps in our computation will depend on this choice, the final result (3.20) is not sensitive to it. We let $\mathcal{U}$ be the superholonomy factor associated to C, defined as in (3.16).

4.1 Preparations

Before proceeding with the computation, we make a few conceptual remarks. First, notice that the object $M(\tau)$ of equation (3.17) involves the Grassmann odd fields ρ_0^i.Technically, the definition of M involves a few steps. Fist, we pick a trivialization $\mathcal{E} = C \times V$ of the pulled-back superbundle over the circle C. Here $V = V_+ \oplus V_-$ is a super-vector space isomorphic with the fiber of E. Then M given in (3.17) can be viewed as an even element of the associative superalgebra $\mathcal{K}_e := \mathcal{F} \otimes End(V)$, where $\mathcal{F}$ is the supercommutative algebra of superfunctions on the circle C. The product in $\mathcal{K}_e$ has the following form on decomposable elements $\phi = \alpha \otimes f$ and $\gamma = \beta \otimes g$:

$$(\alpha \otimes f)(\beta \otimes g) = (-1)^{\mathrm{deg}\beta \mathrm{deg} f}(\alpha\beta) \otimes (f \circ g)\,. \tag{4.1}$$

The total $\mathbb{Z}_2$-valued degree is given by $\deg(\alpha \otimes f) = \deg\alpha + \deg f$.

[4] The construction of [2, 4, 5] was carried out for the abelian case $r_+ = r_- = 1$ and with the supplementary assumption that the target space is $\mathbb{C}^n$, in which case the connections $A^{(\pm)}$ can be gauged away. One also assumed that the target space metric is flat.

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The supertrace (3.8) on $End(E)$ induces the following $\mathcal{F}$-valued trace on $\mathcal{K}_e$:

$$\mathrm{Str}(\alpha \otimes f) = \alpha\, \mathrm{str}(f)\,, \tag{4.2}$$

where $\mathrm{str}(f)$ is of course a complex number, while α is a superfunction on the circle. Str is the supertrace appearing in equation (3.16). Using relation (3.9), one easily checks the cyclicity property:

$$\mathrm{Str}(\phi\gamma) = (-1)^{\deg\phi \deg\gamma}\, \mathrm{Str}(\gamma\phi)\,. \tag{4.3}$$

Viewing superfunctions on the circle as valued in a Grassmann algebra $\mathcal{G}$, the elements of $\mathcal{K}_e$ are valued in the associative superalgebra $K_e := \mathcal{G} \otimes End(V)$. Then the supergroup $GL(r_+|r_-)$ can be viewed as the group of even and invertible elements of K_e (i.e. the group of units of the ordinary associative subalgebra obtained by restricting to the even part of K_e). Even elements of K_e have the form $\phi = \left[\begin{smallmatrix} \phi_{++} & \phi_{-+} \\ \phi_{+-} & \phi_{--} \end{smallmatrix}\right]$, where ϕ_{++} and ϕ_{--} are Grassmann even while ϕ_{+-} and ϕ_{-+} are Grassmann odd. Restricting Str to invertible even elements recovers the supertrace on $GL(r_+|r_-)$.

Let us next give the precise description of the holonomy operator appearing under the supertrace in (3.16). For this, let $\tau \geq \tau_0$ and define even elements $U(\tau, \tau_0)$ of K_e by the formula:[5]

$$U(\tau,\tau_0) = Pe^{-\int_{\tau_0}^{\tau} ds M(s)} = \sum_{n\geq 0} (-1)^n \int_{\tau_0}^{\tau} ds_1 \int_{\tau_0}^{s_1} ds_2 \ldots \int_{\tau_0}^{s_{n-1}} ds_n M(s_1)M(s_2)\ldots M(s_n)\,. \tag{4.4}$$

For $\tau_0 = \tau$ we have $U(\tau_0, \tau_0) = 1$.

Using (4.4), its is easy to check the relation:

$$\frac{\partial}{\partial\tau} U = -M(\tau)U(\tau,\tau_0) \tag{4.5}$$

as well as invertibility of U. Since $M(\tau)$ is periodic with period given by the length l of C, we have:

$$U(\tau + l, \tau_0 + l) = U(\tau, \tau_0)\,. \tag{4.6}$$

Finally, one can use definition (4.4) to check the composition rule:

$$U(\tau_2,\tau_1)U(\tau_1,\tau_0) = U(\tau_2,\tau_0)\,. \tag{4.7}$$

With these preparations, consider the 'holonomy operator':

$$H(\tau) := U(\tau + l, \tau)\,. \tag{4.8}$$

Then the factors $\mathcal{U}$ in (3.16) are defined by:

$$\mathcal{U} = \mathrm{Str}\, H(\tau)\,. \tag{4.9}$$

[5]Technically, we must specify a norm with respect to which the defining series of U is absolutely convergent. This can be achieved by using a Banach Grassmann algebra $\mathcal{G}$ to model the boundary fields. Then K_e becomes a Banach algebra and absolute convergence of the series in (4.4) follows from continuity of M along the compact C, which implies good bounds for the multiple integrals.

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To check that this is independent of τ, notice that:

$$H(\tau) = U(\tau,0)H(0)U(\tau,0)^{-1} \tag{4.10}$$

and use the fact that U is Grassmann even.

We have to clarify one final point. When computing the BRST variation of (3.16), we will need to move the bulk BRST operator over the supertrace. Since the BRST action was originally given only for superfunctions on the worldsheet, this requires that we define an extension δ_e of δ to the associative superalgebra $\mathcal{K}_e = \mathcal{F} \otimes End(V)$. We shall take this extension to be given in the obvious manner, namely:

$$\delta_e := \delta \otimes \mathrm{id}_{End(V)}\,. \tag{4.11}$$

With this definition, δ_e squares to zero and is an odd derivation of the superalgebra $\mathcal{K}_e$. Moreover, we find:

$$\delta\,\mathrm{Str}(\alpha \otimes f) = \delta(\alpha\,\mathrm{str}(f)) = (\delta\alpha)\,\mathrm{str}(f) = \mathrm{Str}(\delta\alpha \otimes f) = \mathrm{Str}(\delta_e(\alpha \otimes f))\,, \tag{4.12}$$

where we recall that $\mathrm{str}(f)$ is a just a complex number. This gives the desired relation:

$$\delta\,\mathrm{Str}(\phi) = \mathrm{Str}(\delta_e\phi) \quad \text{for} \quad \phi \in \mathcal{K}_e\,. \tag{4.13}$$

This construction might seem too trivial to mention, but it has one important consequence. Because we shall compute the BRST variation of $\mathcal{U}$ by working in the algebra $\mathcal{K}_e$, we must treat D as an *odd* element (indeed its $\mathbb{Z}_2$-valued degree in this algebra equals 1). This is true even though D is Grassmann even. For simplicity, we shall denote δ_e by δ from now on.

4.2 The BRST variation of $\mathcal{U}$

We now proceed to compute the BRST variation of $\mathcal{U}$. As for the case of ordinary connections, we find the following formula for the variation of H under an infinitesimal change of M (see appendix B):

$$\delta\,\mathrm{Str}(H(0)) = -\,\mathrm{Str}(H(0)I_C(\delta M))\,, \tag{4.14}$$

where:

$$I_C(\delta M) = \int_0^l d\tau U(\tau)^{-1}\delta M(\tau)U(\tau)\,. \tag{4.15}$$

Here $U(\tau) := U(\tau, 0)$.

Notice that quantity (3.17) can be written as:

$$M = \hat{\mathcal{A}} + \Delta \tag{4.16}$$

where $\hat{\mathcal{A}}d\tau$ is the matrix of the direct sum connection $\mathcal{A} = \left[\begin{smallmatrix}\mathcal{A}^{(+)} & 0\\ & \mathcal{A}^{(-)}\end{smallmatrix}\right]$ on the circle and:

$$\Delta := D^2 + \frac{1}{2}\rho_0^i\partial_i D\,. \tag{4.17}$$

We have:

$$\hat{\mathcal{A}} = \dot{\phi}^{\bar{i}} A_{\bar{i}} + \frac{1}{2} F_{\bar{i}j} \eta^{\bar{i}} \rho_0^j \,. \tag{4.18}$$

Here A is the direct sum connection on $End(E)$ introduced in (3.13).

The BRST variation of M is given by:

$$\delta M = \delta \hat{\mathcal{A}} + \delta \Delta \tag{4.19}$$

where:

$$\delta \hat{\mathcal{A}} = A_{\bar{i}} \dot{\eta}^{\bar{i}} + \partial_{\bar{i}} A_{\bar{j}} \eta^{\bar{i}} \dot{\phi}^{\bar{j}} + F_{i\bar{j}} \dot{\phi}^i \eta^{\bar{j}} + \frac{1}{2} \partial_{\bar{i}} F_{\bar{j}k} \eta^{\bar{i}} \eta^{\bar{j}} \rho_0^k \tag{4.20}$$

and:

$$\delta \Delta = \dot{\phi}^i \partial_i D + \eta^{\bar{i}} \partial_{\bar{i}} (D^2) + \frac{1}{2} \eta^{\bar{i}} \rho_0^j \partial_{\bar{i}} \partial_j D \,. \tag{4.21}$$

Let us write:

$$A_{\bar{i}} \dot{\eta}^{\bar{i}} := \frac{d}{d\tau} (A_{\bar{i}} \eta^{\bar{i}}) - \partial_{\bar{j}} A_{\bar{i}} \eta^{\bar{i}} \dot{\phi}^{\bar{j}} - \partial_i A_{\bar{j}} \dot{\phi}^i \eta^{\bar{j}} \tag{4.22}$$

so that:

$$\delta \hat{\mathcal{A}} = \frac{d}{d\tau} (A_{\bar{i}} \eta^{\bar{i}}) + (\partial_{\bar{i}} A_{\bar{j}} - \partial_{\bar{j}} A_{\bar{i}}) \eta^{\bar{i}} \dot{\phi}^{\bar{j}} + \frac{1}{2} \partial_{\bar{i}} F_{\bar{j}k} \eta^{\bar{i}} \eta^{\bar{j}} \rho_0^k \,. \tag{4.23}$$

To arrive at this relation, we used $F_{i\bar{j}} = \partial_i A_{\bar{j}}$, which holds because $A_i = 0$ for all i (remember that A is a $(0,1)$ connection).

We next notice that:

$$U^{-1} \frac{d}{d\tau} (A_{\bar{i}} \eta^{\bar{i}}) U = \frac{d}{d\tau} (U^{-1} A_{\bar{i}} \eta^{\bar{i}} U) + U^{-1} [A_{\bar{i}} \eta^{\bar{i}}, M] U \,, \tag{4.24}$$

where we used the relations:

$$\frac{d}{d\tau} U = -MU \,, \qquad \frac{d}{d\tau} U^{-1} = U^{-1} M \,. \tag{4.25}$$

In equation (4.24) and below, the symbol $[\cdot,\cdot]$ denotes the usual commutator. Remembering equation (4.16), we find:

$$[A_{\bar{i}} \eta^{\bar{i}}, M] = [A_{\bar{i}} \eta^{\bar{i}}, \hat{\mathcal{A}}] + [A_{\bar{i}} \eta^{\bar{i}}, \Delta] \,, \tag{4.26}$$

with:

$$[A_{\bar{i}} \eta^{\bar{i}}, \hat{\mathcal{A}}] = [A_{\bar{i}}, A_{\bar{j}}] \eta^{\bar{i}} \dot{\phi}^{\bar{j}} + \frac{1}{2} [A_{\bar{i}}, F_{\bar{j}k}] \eta^{\bar{i}} \eta^{\bar{j}} \rho_0^k \tag{4.27}$$

and:

$$[A_{\bar{i}} \eta^{\bar{i}}, \Delta] = \eta^{\bar{i}} [A_{\bar{i}}, D^2] + \frac{1}{2} \eta^{\bar{i}} \rho_0^j [A_{\bar{i}}, \partial_j D] \,. \tag{4.28}$$

Combining (4.23), (4.24) and (4.26) gives:

$$\begin{aligned} U^{-1} \delta \hat{\mathcal{A}} U = {} & \frac{d}{d\tau} (U^{-1} A_{\bar{i}} \eta^{\bar{i}} U) + \\ & + U^{-1} \left((\partial_{\bar{i}} A_{\bar{j}} - \partial_{\bar{j}} A_{\bar{i}}) \eta^{\bar{i}} \dot{\phi}^{\bar{j}} + \frac{1}{2} \partial_{\bar{i}} F_{\bar{j}k} \eta^{\bar{i}} \eta^{\bar{j}} \rho_0^k + [A_{\bar{i}} \eta^{\bar{i}}, \hat{\mathcal{A}}] + [A_{\bar{i}} \eta^{\bar{i}}, \Delta] \right) U \,. \end{aligned} \tag{4.29}$$

Using (4.27) in this expression and combining with (4.19) leads to:

$$U^{-1}\delta M U = \frac{d}{d\tau}(U^{-1}A_{\bar{i}}\eta^{\bar{i}}U) + U^{-1}\left(F_{\bar{i}\bar{j}}\eta^{\bar{i}}\dot{\phi}^{\bar{j}} - \frac{1}{4}\partial_k F_{\bar{i}\bar{j}}\eta^{\bar{i}}\eta^{\bar{j}}\rho_0^k + \delta\Delta + [A_{\bar{i}}\eta^{\bar{i}}, \Delta]\right)U, \quad (4.30)$$

where we used the Bianchi identities for F in order to simplify the second term within the round brackets. Combining (4.21) and (4.28), we find:

$$\delta\Delta + [A_{\bar{i}}\eta^{\bar{i}}, \Delta] = \dot{\phi}^i\partial_i D + \eta^{\bar{i}}\nabla_{\bar{i}}(D^2) + \frac{1}{2}\eta^{\bar{i}}\rho_0^j(\partial_{\bar{i}}\partial_j D + [A_{\bar{i}}, \partial_j D]) \quad (4.31)$$

where $\nabla_{\bar{i}}(D^2) = \partial_{\bar{i}}(D^2) + [A_{\bar{i}}, D^2]$ is the covariant derivative of D^2 with respect to A.

We next want to re-express the term $U^{-1}\dot{\phi}^i\partial_i D U$. Noticing that:

$$\dot{D} = \dot{\phi}^i\partial_i D + \dot{\phi}^{\bar{i}}\partial_{\bar{i}} D\,, \quad (4.32)$$

we obtain:

$$U^{-1}\dot{\phi}^i\partial_i D U = U^{-1}\dot{D}U - U^{-1}\dot{\phi}^{\bar{i}}\partial_{\bar{i}} D U = \frac{d}{d\tau}(U^{-1}DU) + U^{-1}([D, M] - \dot{\phi}^{\bar{i}}\partial_{\bar{i}} D)U\,, \quad (4.33)$$

where again we used equations (4.25). The commutator appearing in the expression above is given by:

$$[D, M] = \dot{\phi}^{\bar{i}}[D, A_{\bar{i}}] + \frac{1}{2}[D, F_{\bar{i}j}]\eta^{\bar{i}}\rho_0^j - \frac{1}{2}\rho_0^i\partial_i(D^2)\,. \quad (4.34)$$

The fact that D anticommutes with ρ_0^i is crucial for obtaining the last term (see the discussion in the previous subsection). Combining with (4.33) gives:

$$U^{-1}\dot{\phi}^i\partial_i D U = \frac{d}{d\tau}(U^{-1}DU) + U^{-1}\left(-\dot{\phi}^{\bar{i}}\nabla_{\bar{i}} D + \frac{1}{2}[D, F_{\bar{i}j}]\eta^{\bar{i}}\rho_0^j - \frac{1}{2}\rho_0^i\partial_i(D^2)\right)U\,. \quad (4.35)$$

We next substitute this into (4.31) to obtain:

$$\begin{aligned} U^{-1}(\delta\Delta + [A_{\bar{i}}\eta^{\bar{i}}, \Delta])U \;=\; & \frac{d}{d\tau}(U^{-1}DU) + \quad (4.36)\\ & + U^{-1}\left(\eta^{\bar{i}}\nabla_{\bar{i}}(D^2) - \dot{\phi}^{\bar{i}}\nabla_{\bar{i}} D - \frac{1}{2}\rho_0^i\partial_i(D^2) + \frac{1}{2}\eta^{\bar{i}}\rho_0^j\partial_j\nabla_{\bar{i}} D\right)U\,. \end{aligned}$$

To arrive at this expression, we used $F_{\bar{i}j} = -\partial_j A_{\bar{i}}$ in (4.35) and combined the second term within the round brackets of that equation with the third term in (4.31) to produce the last term in (4.36).

Finally, we substitute (4.36) into (4.30):

$$\begin{aligned} U^{-1}\delta M U \;=\; & \frac{d}{d\tau}\big[U^{-1}(D + A_{\bar{i}}\eta^{\bar{i}})U\big] + U^{-1}\Big(F_{\bar{i}\bar{j}}\eta^{\bar{i}}\dot{\phi}^{\bar{j}} - \frac{1}{4}\partial_k F_{\bar{i}\bar{j}}\eta^{\bar{i}}\eta^{\bar{j}}\rho_0^k + \eta^{\bar{i}}\nabla_{\bar{i}}(D^2) - \quad (4.37)\\ & - \dot{\phi}^{\bar{i}}\nabla_{\bar{i}} D - \frac{1}{2}\rho_0^i\partial_i(D^2) + \frac{1}{2}\eta^{\bar{i}}\rho_0^j\partial_j\nabla_{\bar{i}} D\Big)U\,. \end{aligned}$$

Using (4.37) in equation (4.15) gives:

$$I_C(\delta M) \;=\; R + \int_0^l d\tau U^{-1}\Big(F_{\bar{i}\bar{j}}\eta^{\bar{i}}\dot{\phi}^{\bar{j}} - \frac{1}{4}\partial_k F_{\bar{i}\bar{j}}\eta^{\bar{i}}\eta^{\bar{j}}\rho_0^k + \eta^{\bar{i}}\nabla_{\bar{i}}(D^2) - \dot{\phi}^{\bar{i}}\nabla_{\bar{i}} D -$$

$$-\frac{1}{2}\rho_0^i\partial_i(D^2)+\frac{1}{2}\eta^{\bar{i}}\rho_0^j\partial_j\nabla_{\bar{i}}D\Big)U\,, \tag{4.38}$$

where $R = H(0)^{-1}(D + A_{\bar{i}}\eta^{\bar{i}})(l)H(0) - (D + A_{\bar{i}}\eta^{\bar{i}})(0)$ is the contribution of the total derivative term. Substituting this into (4.14) leads to relation (3.20) upon noticing that the contribution $\mathrm{Str}(H(0)R)$ vanishes because $H(0)$ is even and due to periodicity of ϕ and η along the boundary:

$$\mathrm{Str}(H(0)R) = \mathrm{Str}\left[(D + A_{\bar{i}}\eta^{\bar{i}})(l)H(0)\right] - \mathrm{Str}\left[H(0)(D + A_{\bar{i}}\eta^{\bar{i}})(0)\right] = 0\,. \tag{4.39}$$

Here $D(l) := D(\phi(l))$ etc.

5. Conclusions

Extending a construction due to [19], we wrote down the general boundary coupling for the B-twisted Landau-Ginzburg model with an arbitrary non-compact Calabi-Yau target space. The open string background is a $(0, 1)$ superconnection living in a complex vector bundle on the target. We also showed that the equations of motion for this background (i.e. the BRST invariance requirement for the partition function on the worldsheet) amount to the condition that the $(0, \leq 2)$ part of the superconnection's curvature equals a constant endomorphism c plus the identity endomorphism multiplied by the Landau-Ginzburg potential W. This is a natural deformation of the target space equations of motion of the B-twisted sigma model, which are recovered in the limit $W = 0$ and $c = 0$ and require vanishing of the $(0, \leq 2)$ part of the curvature. Our results agree with the intuition that the basic branes of the topological Landau-Ginzburg model are condensates of elementary D-branes of the B-model.

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A. Coupling of the B-model to $(0, 1)$ connections

In this appendix we review the coupling of the B-twisted sigma model [21] to ordinary $(0, 1)$ connections as introduced in [19]. Consider the usual B-model on a Riemann surface Σ and focus on a single circle boundary component C. The boundary coupling of [19] has the form:

$$Z_B = \int \mathcal{D}[\phi]\mathcal{D}[\tilde{F}]\mathcal{D}[\theta]\mathcal{D}[\rho]\mathcal{D}[\eta]e^{-\tilde{S}_B}\,\mathrm{Tr}\,H_A \tag{A.1}$$

where:

$$H_A = Pe^{-\oint_C d\tau\hat{\mathcal{A}}} \tag{A.2}$$

is the Wilson loop of the 'improved connection':

$$\hat{\mathcal{A}} = A_{\bar{i}}\dot{\phi}^{\bar{i}} + \frac{1}{2}F_{\bar{i}j}\eta^{\bar{i}}\rho_0^j\,. \tag{A.3}$$

The circle C is the boundary of Σ and A is a $(0,1)$ connection on a complex vector bundle E over the target space X. For the B-twisted sigma model, the bulk action $\tilde{S}_B$ remains BRST closed when considered on bordered Riemann surfaces. Thus BRST invariance of the partition function (A.1) requires:

$$\delta \operatorname{Tr} H_A = 0\,. \tag{A.4}$$

As sketched in [19], the BRST variation of the Wilson loop factor has the form:

$$\delta \operatorname{Tr} H_A(0) = -\operatorname{Tr}\left[H_A(0)\int_0^l d\tau U^{-1}\left(F_{\bar{i}\bar{j}}\eta^{\bar{i}}\dot{\phi}^{\bar{j}} - \frac{1}{4}\partial_k F_{\bar{i}\bar{j}}\eta^{\bar{i}}\eta^{\bar{j}}\rho_0^k\right)U\right] \tag{A.5}$$

where $U(\tau)$ is the parallel transport operator of $\mathcal{A} = \hat{\mathcal{A}}d\tau$ along C, starting from a distinguished point on the boundary which defines the origin of the proper length coordinate τ. Here l is the circumference of τ measured with respect to the metric induced from the interior of Σ. Thus BRST invariance of (A.1) requires that A is an integrable connection.

Let us give the proof of equation (A.5). First notice that $\mathcal{A}$ is a connection on the complex pulled-back bundle $\mathcal{E} = \phi_\partial^*(E)$, which can be trivialized over C (here ϕ_∂ is the restriction of ϕ to the boundary $\partial\Sigma$). Hence we can view $\hat{\mathcal{A}}$ as a matrix-valued function on the circle. Thus $U(\tau)$ is uniquely determined by the equation:

$$\left(\frac{d}{d\tau} + \hat{\mathcal{A}}\right)U = 0 \tag{A.6}$$

and the initial condition $U(0) = \mathrm{id}$. The holonomy operator at the origin is given by $H_A(0) := U(l)$, where l is the length of C. Varying the origin changes $H_A(0)$ by a similarity transformation, but does not affect its trace.

Let us consider the change of $H_A(0)$ induced by an arbitrary variation of $\mathcal{A}$. Taking the variation of (A.6) gives:

$$\left(\frac{d}{d\tau} + \hat{\mathcal{A}}\right)\delta U = -\delta\hat{\mathcal{A}}U\,. \tag{A.7}$$

This is equivalent with:

$$\frac{d}{d\tau}\Phi = -U^{-1}\delta\hat{\mathcal{A}}U \tag{A.8}$$

where we introduced the quantity $\Phi := U^{-1}\delta U$. The initial condition $U(0) = \mathrm{id}$ gives $\delta U(0) = 0$ and thus $\Phi(0) = 0$. With this constraint, equation (A.8) is solved by:

$$\Phi(\tau) = -\int_0^\tau ds U(s)^{-1}\delta\hat{\mathcal{A}}(s)U(s) \tag{A.9}$$

so that:

$$\delta U(\tau) = -U(\tau)\int_0^\tau ds U(s)^{-1}\delta\hat{\mathcal{A}}(s)U(s)\,. \tag{A.10}$$

This gives:

$$\operatorname{Tr}\delta H_A(0) = -\operatorname{Tr}\left[H_A(0)\int_0^l d\tau U(\tau)^{-1}\delta\hat{\mathcal{A}}(\tau)U(\tau)\right]\,. \tag{A.11}$$

It is easy to check that the right hand side is independent of the choice of origin for τ.

To recover (A.5), we must apply this formula for the BRST variation of $\hat{\mathcal{A}}$. The BRST transformations of the B-twisted sigma model are given by (2.3) with W set to zero. We have:

$$\delta\hat{\mathcal{A}} = A_{\bar{i}}\dot{\eta}^{\bar{i}} + \partial_{\bar{i}}A_{\bar{j}}\eta^{\bar{i}}\dot{\phi}^{\bar{j}} + F_{i\bar{j}}\dot{\phi}^{i}\eta^{\bar{j}} + \frac{1}{2}\partial_{\bar{i}}F_{\bar{j}k}\eta^{\bar{i}}\eta^{\bar{j}}\rho_0^k\,, \tag{A.12}$$

where the dot stands for $\frac{d}{d\tau}$.

To eliminate the τ-derivative of η in the first term, we write:

$$U^{-1}A_{\bar{i}}\dot{\eta}^{\bar{i}}U = U^{-1}\frac{d}{d\tau}(A_{\bar{i}}\eta^{\bar{i}})U - U^{-1}\left[\partial_i A_{\bar{j}}\dot{\phi}^i\eta^{\bar{j}} + \partial_{\bar{j}}A_{\bar{i}}\eta^{\bar{i}}\dot{\phi}^{\bar{j}}\right]U \tag{A.13}$$

and:

$$U^{-1}\frac{d}{d\tau}(A_{\bar{i}}\eta^{\bar{i}})U = \frac{d}{d\tau}(U^{-1}A_{\bar{i}}\eta^{\bar{i}}U) + U^{-1}[A_{\bar{i}}\eta^{\bar{i}},\hat{\mathcal{A}}]U\,, \tag{A.14}$$

where we used the relations:

$$\frac{d}{d\tau}U = -\hat{\mathcal{A}}U\,, \qquad \frac{d}{d\tau}U^{-1} = U^{-1}\hat{\mathcal{A}}\,. \tag{A.15}$$

The commutator in (A.14) is given by:

$$[A_{\bar{i}}\eta^{\bar{i}},\hat{\mathcal{A}}] = [A_{\bar{i}},A_{\bar{j}}]\eta^{\bar{i}}\dot{\phi}^{\bar{j}} + \frac{1}{2}[A_{\bar{i}},F_{\bar{j}k}]\eta^{\bar{i}}\eta^{\bar{j}}\rho_0^k\,. \tag{A.16}$$

Combining everything, we find:

$$U^{-1}\delta\hat{\mathcal{A}}U = \frac{d}{d\tau}(U^{-1}A_{\bar{i}}\eta^{\bar{i}}U) + U^{-1}\left[F_{\bar{i}\bar{j}}\eta^{\bar{i}}\dot{\phi}^{\bar{j}} - \frac{1}{4}\partial_k F_{\bar{i}\bar{j}}\eta^{\bar{i}}\eta^{\bar{j}}\rho_0^k\right]U\,, \tag{A.17}$$

where

$$F_{\bar{i}\bar{j}} = \partial_{\bar{i}}A_{\bar{j}} - \partial_{\bar{j}}A_{\bar{i}} + [A_{\bar{i}},A_{\bar{j}}] \tag{A.18}$$

is the $(0,2)$ part of the curvature of A. To arrive at (A.17), we used the Bianchi identities and the relation $F_{i\bar{j}} = \partial_i A_{\bar{j}}$ (which holds because A is a $(0,1)$ connection). Using (A.17) into (A.11) leads to (A.5) upon noticing that the boundary term induced by the total derivative in (A.17) brings vanishing contribution to (A.11) due to the periodicity of ϕ and η along the boundary.

B. Variation of H with respect to M

In this appendix we derive relation (4.14), which gives the infinitesimal change of the 'holonomy' operator H under a variation of M. We shall use the notation $U(\tau) := U(\tau,0)$ as in section 4. The argument is very similar to that of appendix A.

Remember from (4.25) that $U(\tau)$ satisfies:

$$\left(\frac{d}{d\tau} + M\right)U = 0 \tag{B.1}$$

with the initial condition $U(0) = \mathrm{id}$.

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Consider the change of $H(0)$ induced by a variation of M. Taking the variation of (B.1) gives:

$$\left(\frac{d}{d\tau} + M\right) \delta U = -\delta M U \,. \tag{B.2}$$

This is equivalent with:

$$\frac{d}{d\tau}\Phi = -U^{-1}\delta M U \tag{B.3}$$

where we introduced $\Phi := U^{-1}\delta U$. The constraint $U(0) = \mathrm{id}$ gives $\delta U(0) = 0$ and thus $\Phi(0) = 0$. With this initial condition, equation (A.8) is solved by:

$$\Phi(\tau) = -\int_0^\tau ds U(s)^{-1}\delta M(s) U(s) \,, \tag{B.4}$$

which gives:

$$\delta U(\tau) = -U(\tau)\int_0^\tau ds U(s)^{-1}\delta M(s) U(s) \,. \tag{B.5}$$

Applying this for $\tau = l$ and recalling that $U(l) = U(l, 0) = H(0)$, we find:

$$\delta H(0) = -H(0)\int_0^l ds U(s)^{-1}\delta M(s) U(s) \,. \tag{B.6}$$

This implies equation (4.14).

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